Solution to Inner Product Space Problem Set

- Tutor: Nelson Lam (nelson.lam.lcy@gmail.com)

- All the suggestions and feedback are welcome. Any report of typos is appreciated.

1 In Class Exercise

Exercise 1.1. Let $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = -\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$

(a). Compute $\|\mathbf{u}\|, \|\mathbf{v}\|, \langle \mathbf{u}, \mathbf{v} \rangle$

(b). Compute the angle between $\mathbf{u}$ and $\mathbf{v}$

Solution.

$$\|\mathbf{u}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\|\mathbf{v}\| = \sqrt{(-1)^2 + 5^2 + (-3)^2} = \sqrt{35}$$

$$\langle \mathbf{u}, \mathbf{v} \rangle = (1)(-1) + (1)(5) + (1)(-3) = 1$$

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{1}{\sqrt{3}\sqrt{35}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{105}}\right)$$

Exercise 1.2. Let $\mathbf{u} = 8\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$

(a). Compute $\|\mathbf{u}\|, \|\mathbf{v}\|, \langle \mathbf{u}, \mathbf{v} \rangle$

(b). Compute $\mathrm{Proj}_{\mathbf{v}}\mathbf{u}$

(c). Write $\mathbf{u}$ as the sum of a vector parallel to $\mathbf{v}$ and orthogonal to $\mathbf{v}$

Solution.

$$\|\mathbf{u}\| = \sqrt{8^2 + 4^2 + (-12)^2} = \sqrt{224} = 4\sqrt{14}$$

$$\|\mathbf{v}\| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$$

$$\langle \mathbf{u}, \mathbf{v} \rangle = (8)(1) + (4)(2) + (-12)(-1) = 28$$

$$\mathrm{Proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{28}{6}(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = \frac{14}{3}\mathbf{i} + \frac{28}{3}\mathbf{j} - \frac{14}{3}\mathbf{k}$$

Let $\mathbf{w} = \mathbf{u} - \mathrm{Proj}_{\mathbf{v}}\mathbf{u} = \frac{10}{3}\mathbf{i} - \frac{16}{3}\mathbf{j} - \frac{22}{3}\mathbf{k}$, then $\mathbf{u} = \mathbf{w} + \mathrm{Proj}_{\mathbf{v}}\mathbf{u}$

$$\langle \mathbf{w}, \mathbf{v} \rangle = \left(\frac{10}{3}\right)(1) + \left(-\frac{16}{3}\right)(2) + \left(-\frac{22}{3}\right)(-1) = 0$$

By construction, $\mathrm{Proj}_{\mathbf{v}}\mathbf{u}$ is a vector parallel to $\mathbf{v}$. While $\mathbf{w}$ is a vector orthogonal to $\mathbf{u}$

Exercise 1.3 (Lecture Notes Chapter 1 Q2)**.** Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ with $\langle \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle, \forall\, \mathbf{w} \in \mathbb{R}^3$

(a). Show that $\langle \mathbf{u} - \mathbf{v}, \mathbf{w} \rangle = 0$, for any choice of $\mathbf{w}$

(b). Use the fact that $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Rightarrow \mathbf{x} = \mathbf{0}$. Cleverly choose a vector $\mathbf{w}$ and substitute into (a). Conclude that $\mathbf{u} = \mathbf{v}$

Solution. Suppose $\langle \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle, \forall\, \mathbf{w} \in \mathbb{R}^3$, then

$$\begin{aligned} \langle \mathbf{u}, \mathbf{w} \rangle - \langle \mathbf{v}, \mathbf{w} \rangle &= 0 \\ \langle \mathbf{u} - \mathbf{v}, \mathbf{w} \rangle &= 0 \end{aligned}$$

Since $\mathbf{w}$ is arbitrary (to be chosen), choose $\mathbf{w} = \mathbf{u} - \mathbf{v}$, then

$$\begin{aligned} \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle &= 0 \\ \mathbf{u} - \mathbf{v} &= \mathbf{0} \\ \mathbf{u} &= \mathbf{v} \end{aligned}$$

Exercise 1.4 (2020 TDG Quiz 1 Q1 Modified)**.**
Let $\mathbf{u} = \frac{\sqrt{6}}{6}\,\mathbf{i} + \frac{\sqrt{6}}{6}\,\mathbf{j} + \frac{\sqrt{6}}{3}\,\mathbf{k}, \mathbf{v} = \frac{\sqrt{2}}{2}\,\mathbf{i} - \frac{\sqrt{2}}{2}\,\mathbf{j}, \mathbf{w} = \frac{\sqrt{3}}{3}\,\mathbf{i} + \frac{\sqrt{3}}{3}\,\mathbf{j} - \frac{\sqrt{3}}{3}\,\mathbf{k}$ be 3 vectors in $\mathbb{R}^3$

(a). Show that they are mutually orthogonal

(b). Using (a), show that they constitutes an orthonormal basis for $\mathbb{R}^3$

(c). Express $\mathbf{r} = 2024\,\mathbf{i} + 8\,\mathbf{j} + 12\,\mathbf{k}$ as a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

Solution.

$$\langle \mathbf{u}, \mathbf{v} \rangle = \left(\frac{\sqrt{6}}{6}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{6}}{6}\right)\left(-\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{6}}{3}\right)(0) = 0$$

$$\langle \mathbf{u}, \mathbf{w} \rangle = \left(\frac{\sqrt{6}}{6}\right)\left(\frac{\sqrt{3}}{3}\right) + \left(\frac{\sqrt{6}}{6}\right)\left(\frac{\sqrt{3}}{3}\right) + \left(\frac{\sqrt{6}}{3}\right)\left(-\frac{\sqrt{3}}{3}\right) = 0$$

$$\langle \mathbf{v}, \mathbf{w} \rangle = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{3}\right) + \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{3}\right) + (0)\left(-\frac{\sqrt{3}}{3}\right) = 0$$

$$\|\mathbf{u}\| = \sqrt{\left(\frac{\sqrt{6}}{6}\right)^2 + \left(\frac{\sqrt{6}}{6}\right)^2 + \left(\frac{\sqrt{6}}{3}\right)^2} = 1$$

$$\|\mathbf{v}\| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2 + (0)^2} = 1$$

$$\|\mathbf{w}\| = \sqrt{\left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{3}}{3}\right)^2 + \left(-\frac{\sqrt{3}}{3}\right)^2} = 1$$

Suppose $\mathbf{r} = \alpha\,\mathbf{u} + \beta\,\mathbf{v} + \gamma\,\mathbf{w}$, then:

$$\alpha = \langle \mathbf{r}, \mathbf{u} \rangle = (2024)\left(\frac{\sqrt{6}}{6}\right) + (8)\left(\frac{\sqrt{6}}{6}\right) + (12)\left(\frac{\sqrt{6}}{3}\right) = \frac{1028\sqrt{6}}{3}$$

$$\beta = \langle \mathbf{r}, \mathbf{v} \rangle = (2024)\left(\frac{\sqrt{2}}{2}\right) + (8)\left(-\frac{\sqrt{2}}{2}\right) + 0 = 1008\sqrt{2}$$

$$\gamma = \langle \mathbf{r}, \mathbf{w} \rangle = (2024)\left(\frac{\sqrt{3}}{3}\right) + (8)\left(\frac{\sqrt{3}}{3}\right) + (12)\left(-\frac{\sqrt{3}}{3}\right) = \frac{2020\sqrt{3}}{3}$$

Exercise 1.5. Let $\mathbf{u} = \frac{\sqrt{6}}{6}\mathbf{i} + \frac{\sqrt{6}}{6}\mathbf{j} + \frac{\sqrt{6}}{3}\mathbf{k}, \mathbf{v} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$

(a). Compute $\mathbf{u} \times \mathbf{v}$, is it a unit vector ?

(b). "If $\mathbf{w}$ is not given in Exercise 1.4, then there is insufficient data to finish Exercise 1.3", do you agree ?

Solution.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{vmatrix} = \frac{\sqrt{3}}{3}\mathbf{i} + \frac{\sqrt{3}}{3}\mathbf{j} - \frac{\sqrt{3}}{3}\mathbf{k}$$

$$\left\| \frac{\sqrt{3}}{3}\mathbf{i} + \frac{\sqrt{3}}{3}\mathbf{j} - \frac{\sqrt{3}}{3}\mathbf{k} \right\| = \sqrt{\left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{3}}{3}\right)^2 + \left(-\frac{\sqrt{3}}{3}\right)^2} = 1$$

Disagree, examiners can **hell** you in Exercise 1.4 by omitting $\mathbf{w}$, but give 1 marks for derivation. If you can't derive $\mathbf{w}$, then you probably get zero in whole question (domino effect !)

Warm Reminder: Make sure you grasp basic techniques to secure "easy" scores

Exercise 1.6 (2023 TDG Quiz 1 Q2)**.**

(a). Evaluate the following determinants:

$$\det(R_z) = \det \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det(R_y) = \det \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$\det(R_x) = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix}$$

(b). Describe the geometric meaning of the three matrices R_x, R_y, R_z above.

(c). Describe the geometric meaning of the product $R_xR_yR_z$ of the three matrices above.

Solution.

(a). $\det(R_z) = \cos^2\alpha + \sin^2\alpha = \det(R_y) = \cos^2\beta + \sin^2\beta = \det(R_x) = \cos^2\gamma + \sin^2\gamma = 1$

(b). By matrix representation, consider R_z

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Hence R_z represents rotation about $z-$axis by α anticlockwisely

Similarly, R_y represents rotation about $y-$axis by β anticlockwisely and R_x represents rotation about $x-$axis by γ anticlockwisely

(c). First rotate about $z-$axis by α anticlockwisely, then rotate about $y-$axis by β anticlockwisely. After that, rotate about $x-$axis by γ anticlockwisely

Exercise 1.7. Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ be a regular parameterized plane curve, with $\|\mathbf{r}'(s)\| = 1$, then:

(a). Prove that $\langle \mathbf{r}'(s), \mathbf{r}''(s) \rangle = 0$

(b). Conclude that $\mathbf{r}'(s) \perp \mathbf{r}''(s)$

(c). What is the direction of $\mathbf{r}'(s)$ and $\mathbf{r}''(s)$ with respect to $\mathbf{r}(s)$?

Solution (Solution to (a)).

$$\begin{aligned}
\|\mathbf{r}'(s)\| &= 1 \\
\langle \mathbf{r}'(s), \mathbf{r}'(s) \rangle &= 1 \\
\frac{d}{ds}\langle \mathbf{r}'(s), \mathbf{r}'(s) \rangle &= \frac{d}{ds}(1) \\
\langle \mathbf{r}'(s), \mathbf{r}''(s) \rangle + \langle \mathbf{r}''(s), \mathbf{r}'(s) \rangle &= 0 \\
2\langle \mathbf{r}'(s), \mathbf{r}''(s) \rangle &= 0 \\
\langle \mathbf{r}'(s), \mathbf{r}''(s) \rangle &= 0
\end{aligned}$$

Solution (Solution to (b)).
By (a), $\langle \mathbf{r}'(s), \mathbf{r}''(s) \rangle = 0$, hence by definition of orthogonal, $\mathbf{r}'(s) \perp \mathbf{r}''(s)$

Solution (Solution to (c)).
Suppose Γ is the trace of $\mathbf{r}(s)$. Let s_0 be fixed, then $\mathbf{r}(s_0)$ is a point on Γ

- $\mathbf{r}'(s_0)$ is the vector pointing at tangential direction of Γ at $\mathbf{r}(s_0)$
- $\mathbf{r}''(s_0)$ is the vector pointing at the normal direction of Γ at $\mathbf{r}(s_0)$

Hence $\mathbf{r}'(s_0)$ and $\dfrac{\mathbf{r}''(s_0)}{\|\mathbf{r}''(s_0)\|}$ is a frame (unit square) indicating the curvature of Γ

2 Warm up Question

Exercise 2.1. Let $\mathbf{u} = \mathbf{i}+\mathbf{j}+\mathbf{k}, \mathbf{v} = \mathbf{i}-\mathbf{j}, \mathbf{w} = \mathbf{i}+\mathbf{j}-2\,\mathbf{k}$. Suppose $\mathbf{r} = \mathbf{i}+2\,\mathbf{j}+3\,\mathbf{k}$, find $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\mathbf{r} = \alpha\,\mathbf{u}+\beta\,\mathbf{v}+\gamma\,\mathbf{w}$

Solution. Notice that $\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{w}, \mathbf{u}\rangle = 0$

Hence $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ constitutes an orthonormal basis, suppose

$$
\begin{aligned}
\mathbf{r} &= \alpha\,\mathbf{u}+\beta\,\mathbf{v}+\gamma\,\mathbf{w} \\
\langle \mathbf{r}, \mathbf{u}\rangle &= \alpha\langle \mathbf{u}, \mathbf{u}\rangle + \beta\langle \mathbf{v}, \mathbf{u}\rangle + \gamma\langle \mathbf{w}, \mathbf{u}\rangle \\
(1)(1) + (2)(1) + (3)(1) &= \alpha(1^2 + 1^2 + 1^2) + 0 + 0 \\
\alpha &= 2 \\
\langle \mathbf{r}, \mathbf{v}\rangle &= \alpha\langle \mathbf{u}, \mathbf{v}\rangle + \beta\langle \mathbf{v}, \mathbf{v}\rangle + \gamma\langle \mathbf{w}, \mathbf{v}\rangle \\
(1)(1) + (2)(-1) + (3)(0) &= 0 + \beta(1^2 + (-1)^2) + 0 \\
\beta &= -\frac{1}{2} \\
\langle \mathbf{r}, \mathbf{w}\rangle &= \alpha\langle \mathbf{u}, \mathbf{w}\rangle + \beta\langle \mathbf{v}, \mathbf{w}\rangle + \gamma\langle \mathbf{w}, \mathbf{w}\rangle \\
(1)(1) + (2)(1) + (3)(-2) &= 0 + 0 + \gamma(1^2 + 1^2 + (-2)^2) \\
\gamma &= -\frac{1}{2}
\end{aligned}
$$

Exercise 2.2. Let $\mathbf{u} = \mathbf{i}+\mathbf{j}+\mathbf{k}, \mathbf{v} = \mathbf{i}-\mathbf{j}$ be two vectors in $\mathbb{R}^3$

(a). Is $\mathbf{u}$ orthogonal to $\mathbf{v}$?

(b). Compute $\mathbf{u}\times\mathbf{v}$

(c). Using (a), (b), construct an orthonormal basis for $\mathbb{R}^3$

Solution.

$$\langle \mathbf{u}, \mathbf{v}\rangle = (1)(1) + (1)(-1) + (1)(0) = 0$$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \mathbf{i}+\mathbf{j}-2\,\mathbf{k}$$

By previous computation, $\{\mathbf{i}+\mathbf{j}+\mathbf{k}, \mathbf{i}-\mathbf{j}, \mathbf{i}+\mathbf{j}-2\,\mathbf{k}\}$ is a orthogonal basis
It remains to normalize each vector (divide each by their length)
Hence the orthonormal basis is $\{\frac{1}{\sqrt{3}}\,\mathbf{i}+\frac{1}{\sqrt{3}}\,\mathbf{j}+\frac{1}{\sqrt{3}}\,\mathbf{k}, \frac{1}{\sqrt{2}}\,\mathbf{i}-\frac{1}{\sqrt{2}}\,\mathbf{j}, \frac{1}{\sqrt{6}}\,\mathbf{i}+\frac{1}{\sqrt{6}}\,\mathbf{j}-\frac{2}{\sqrt{6}}\,\mathbf{k}\}$

Exercise 2.3 (2015 TDG Quiz 1 Q1)**.**
Let $\mathbf{u} = (1, 1, 0)$ and $\mathbf{v} = (3, 6, 9)$ be two vectors in $\mathbb{R}^3$

(a). Let $\mathbf{w} = \mathbf{v} - \dfrac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|^2}\,\mathbf{u}$, compute $\mathbf{w}$

(b). Show that $\mathbf{u}, \mathbf{w}$ are orthogonal

Solution.

$$\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{i} + \mathbf{j}, 3\,\mathbf{i} + 6\,\mathbf{j} + 9\,\mathbf{k}\rangle = (1)(3) + (1)(6) = 9$$

$$\|\mathbf{u}\|^2 = \|\mathbf{i} + \mathbf{j}\|^2 = 1^2 + 1^2 = 2$$

$$\mathbf{w} = \mathbf{v} - \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|^2}\,\mathbf{u} = 3\,\mathbf{i} + 6\,\mathbf{j} + 9\,\mathbf{k} - \frac{9}{2}(\,\mathbf{i} + \mathbf{j}\,) = -\frac{3}{2}\,\mathbf{i} + \frac{3}{2}\,\mathbf{j} + 9\,\mathbf{k}$$

$$\langle \mathbf{u}, \mathbf{w}\rangle = (1)\left(-\frac{3}{2}\right) + (1)\left(\frac{3}{2}\right) + (0)(9) = 0$$

Hence $\mathbf{u}, \mathbf{w}$ are orthogonal

Exercise 2.4. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be an orthogonal subset of $\mathbb{R}^m$. Suppose:

$$\mathbf{v} = \alpha_1\,\mathbf{v}_1 + \cdots + \alpha_k\,\mathbf{v}_k$$

$$\mathbf{w} = \beta_1\,\mathbf{v}_1 + \cdots + \beta_k\,\mathbf{v}_k$$

Show that for $\alpha_i, \beta_i \in \mathbb{R}, i = 1, 2, \ldots, k$, we have:

$$\langle \mathbf{v}, \mathbf{w}\rangle = \alpha_1\beta_1\|\mathbf{v}_1\|^2 + \cdots + \alpha_k\beta_k\|\mathbf{v}_k\|^2$$

Solution.

$$\begin{aligned}
\langle \mathbf{v}, \mathbf{w}\rangle &= \langle \alpha_1\,\mathbf{v}_1 + \cdots + \alpha_k\,\mathbf{v}_k, \beta_1\,\mathbf{v}_1 + \cdots + \beta_k\,\mathbf{v}_k\rangle \\
&= \left\langle \sum_{i=1}^{k} \alpha_i\,\mathbf{v}_i, \sum_{j=1}^{k} \beta_j\,\mathbf{v}_j \right\rangle \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k} \langle \alpha_i\,\mathbf{v}_i, \beta_j\,\mathbf{v}_j\,\rangle \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\beta_j\langle\,\mathbf{v}_i, \mathbf{v}_j\,\rangle \\
&= \sum_{i,j=1}^{k} \alpha_i\beta_j\langle\,\mathbf{v}_i, \mathbf{v}_j\,\rangle \\
&= \sum_{i=1}^{k} \alpha_i\beta_i\langle\,\mathbf{v}_i, \mathbf{v}_i\,\rangle + \sum_{i\neq j \text{ and } i,j=1}^{k} \alpha_i\beta_j\langle\,\mathbf{v}_i, \mathbf{v}_j\,\rangle
\end{aligned}$$

Since $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be an orthogonal subset of $\mathbb{R}^m$, hence $\langle \mathbf{v}_i, \mathbf{v}_j\rangle = 0$ for $i \neq j$, hence

$$\begin{aligned}
\langle \mathbf{v}, \mathbf{w}\rangle &= \sum_{i=1}^{k} \alpha_i\beta_i\langle\,\mathbf{v}_i, \mathbf{v}_i\,\rangle + \sum_{i\neq j \text{ and } i,j=1}^{k} \alpha_i\beta_j\langle\,\mathbf{v}_i, \mathbf{v}_j\,\rangle \\
&= \sum_{i=1}^{k} \alpha_i\beta_i\langle\,\mathbf{v}_i, \mathbf{v}_i\,\rangle + 0 \\
&= \alpha_1\beta_1\|\mathbf{v}_1\|^2 + \cdots + \alpha_k\beta_k\|\mathbf{v}_k\|^2
\end{aligned}$$

Exercise 2.5 (2001 HKALE Pure Math Paper 1 Q7)**.**
A 2×2 matrix M is the matrix representation of a transformation T in $\mathbb{R}^2$. It is known that T transforms $(1,0)$ and $(0,1)$ to $(1,1)$ and $(-1,1)$ respectively.

(a). Find M

(b). Find $\lambda > 0$ and $a, b, c, d \in \mathbb{R}$ such that $ad - bc = 1$ and M can be decomposed as:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Hence describe the geometric meaning of T

Solution.

$$M = \left(T\begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle| T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Notice that

$$\det\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \lambda^2(ad - bc) = \lambda^2$$

By comparison, $\det(M) = 2$, hence $\lambda = \sqrt{2}$, then we write

$$M = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{pmatrix}$$

The matrix M represents a linear transformation which: First rotate by 45° anticlockwise, then enlarge by $\sqrt{2}$ times

Exercise 2.6. For the following vector-valued functions, compute their derivatives:

(a). $\mathbf{r}(t) = (t^2 + 1, 2t, 9 - t)$

(b). $\mathbf{r}(t) = 3t[3t\,\mathbf{i} - 9t^2\,\mathbf{j} + (\cos t)\,\mathbf{k}]$

(c). $\mathbf{r}(t) = [(\sin t)\,\mathbf{i} - (\cos t)\,\mathbf{j}] \times [4t^7\,\mathbf{i} - 3t^2\,\mathbf{j} + (t^3 - t^2)\,\mathbf{k}]$

Solution. (a). $\mathbf{r}'(t) = (2t, 2, -1)$

(b). $\mathbf{r}'(t) = (18t, -81t^2, -3t\sin t + 3\cos t)$

(c).

$$\mathbf{r}(t) = [(\sin t)\,\mathbf{i} - (\cos t)\,\mathbf{j}] \times [4t^7\,\mathbf{i} - 3t^2\,\mathbf{j} + (t^3 - t^2)\,\mathbf{k}] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin t & -\cos t & 0 \\ 4t^7 & -3t^2 & t^3 - t^2 \end{vmatrix}$$

$$\begin{aligned} \mathbf{r}(t) &= (t^3 - t^2)\cos t\,\mathbf{i} + (t^2 - t^3)\sin t\,\mathbf{j} + (-3t^2\sin t + 4t^7\cos t)\,\mathbf{k} \\ \mathbf{r}'(t) &= t((-2 + 3t)\cos t - (-1 + t)t\sin t)\,\mathbf{i} + -t((-1 + t)t\cos t + (-2 + 3t)\sin t)\,\mathbf{j} \\ &\quad - t((3t - 28t^5)\cos t + 2(3 + 2t^6)\sin t)\,\mathbf{k} \end{aligned}$$

3 Standard Question

3.1 Orthogonalization & Orthogonal Basis

Exercise 3.1 (HKALE 1993 Pure Math Paper 1 Q8)**.**
Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be linearly independent vectors in $\mathbb{R}^3$, show that:

(a). Suppose $\mathbf{s} \in \mathbb{R}^3$ with $\langle \mathbf{s}, \mathbf{u} \rangle = \langle \mathbf{s}, \mathbf{v} \rangle = \langle \mathbf{s}, \mathbf{w} \rangle = 0$, then $\mathbf{s} = \mathbf{0}$

(b). Suppose $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = 0$, prove that $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{u} \rangle = 0$

Deduce that: $\forall\, \mathbf{r} \in \mathbb{R}^3, \mathbf{r} = \dfrac{\langle \mathbf{r}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u} + \dfrac{\langle \mathbf{r}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v} + \dfrac{\langle \mathbf{r}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle}\mathbf{w}$

Solution (Solution to (a)). Let $s = (s_1, s_2, s_3), u = (u_1, u_2, u_3), v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$
Then we can form an equation in (s_1, s_2, s_3), that is:

$$(E) : \begin{cases} u_1 s_1 + u_2 s_2 + u_3 s_3 = 0 \\ v_1 s_1 + v_2 s_2 + v_3 s_3 = 0 \\ w_1 s_1 + w_2 s_2 + w_3 s_3 = 0 \end{cases}$$

By assumption, $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, hence:

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} \neq 0$$

Since $\det(A) = \det(A^T)$ for any 3×3 matrix, hence:

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \neq 0$$

Therefore the following homogenous system has unique solution

$$\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence (E) has unique solution $(s_1, s_2, s_3) = (0, 0, 0)$, thus $\mathbf{s} = \mathbf{0}$

Solution (Solution to (b)). Suppose $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = 0$, then $\mathbf{u} = \mu(\mathbf{v} \times \mathbf{w})$ for some $\mu \in \mathbb{R}$

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mu \langle \mathbf{v} \times \mathbf{w}, \mathbf{v} \rangle = 0$$
$$\langle \mathbf{u}, \mathbf{w} \rangle = \mu \langle \mathbf{v} \times \mathbf{w}, \mathbf{w} \rangle = 0$$

Similarly, suppose $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = 0$, then $\mathbf{w} = \lambda(\mathbf{u} \times \mathbf{v})$ for some $\lambda \in \mathbb{R}$

$$\langle \mathbf{w}, \mathbf{v} \rangle = \lambda \langle \mathbf{u} \times \mathbf{v}, \mathbf{v} \rangle = 0$$

Hence $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{u} \rangle = 0$

Solution (Solution to (c)). Suppose $\mathbf{r} = \alpha\,\mathbf{u} + \beta\,\mathbf{v} + \gamma\,\mathbf{w}$

$$
\begin{aligned}
\langle \mathbf{r}, \mathbf{u} \rangle &= \alpha \langle \mathbf{u}, \mathbf{u} \rangle + \beta \langle \mathbf{v}, \mathbf{u} \rangle + \gamma \langle \mathbf{w}, \mathbf{u} \rangle \\
\langle \mathbf{r}, \mathbf{u} \rangle &= \alpha \langle \mathbf{u}, \mathbf{u} \rangle + 0 + 0 \\
\alpha &= \frac{\langle \mathbf{r}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \\
\langle \mathbf{r}, \mathbf{v} \rangle &= \alpha \langle \mathbf{u}, \mathbf{v} \rangle + \beta \langle \mathbf{v}, \mathbf{v} \rangle + \gamma \langle \mathbf{w}, \mathbf{v} \rangle \\
\langle \mathbf{r}, \mathbf{v} \rangle &= 0 + \beta \langle \mathbf{v}, \mathbf{v} \rangle + 0 \\
\beta &= \frac{\langle \mathbf{r}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \\
\langle \mathbf{r}, \mathbf{w} \rangle &= \alpha \langle \mathbf{u}, \mathbf{w} \rangle + \beta \langle \mathbf{v}, \mathbf{w} \rangle + \gamma \langle \mathbf{w}, \mathbf{w} \rangle \\
\langle \mathbf{r}, \mathbf{u} \rangle &= 0 + 0 + \gamma \langle \mathbf{u}, \mathbf{w} \rangle \\
\gamma &= \frac{\langle \mathbf{r}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle}
\end{aligned}
$$

Therefore $\forall\,\mathbf{r} \in \mathbb{R}^3, \mathbf{r} = \dfrac{\langle \mathbf{r}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\,\mathbf{u} + \dfrac{\langle \mathbf{r}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\,\mathbf{v} + \dfrac{\langle \mathbf{r}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle}\,\mathbf{w}$

Exercise 3.2 (2003 HKALE Pure Math Paper 1 Q9).
Consider the vectors $\mathbf{a} = (p, q, 0), \mathbf{b} = (q, -p, 0), \mathbf{c} = (0, 0, r)$, where $p, q, r \in \mathbb{R}\backslash\{0\}$

(a). Prove that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly independent

(b). Let $\mathbf{d} \in \mathbb{R}^3$, prove that $\mathbf{d} = \dfrac{\langle \mathbf{d}, \mathbf{a} \rangle}{\langle \mathbf{a}, \mathbf{a} \rangle} \mathbf{a} + \dfrac{\langle \mathbf{d}, \mathbf{b} \rangle}{\langle \mathbf{b}, \mathbf{b} \rangle} \mathbf{b} + \dfrac{\langle \mathbf{d}, \mathbf{c} \rangle}{\langle \mathbf{c}, \mathbf{c} \rangle} \mathbf{c}$

(c). Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$ are linearly independent, define $\mathbf{u} = \mathbf{x}$ and $\mathbf{v} = \mathbf{y} - \dfrac{\langle \mathbf{y}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}$

Prove that $\mathbf{v}$ is a non-zero vector

(d). Define $\mathbf{w} = \mathbf{z} - \dfrac{\langle \mathbf{z}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u} - \dfrac{\langle \mathbf{z}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}$

- Prove that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are orthogonal
- Describe the geometric relationship between $\mathbf{w}$ and the plane containing the vectors $\mathbf{x}$ and $\mathbf{y}$

Solution (Solution to (a)).
Suppose that $\alpha\,\mathbf{a} + \beta\,\mathbf{b} + \gamma\,\mathbf{c} = \mathbf{0}$ for some $\alpha, \beta, \gamma \in \mathbb{R}$
Then $\alpha(p, q, 0) + \beta(q, -p, 0) + \gamma(0, 0, r) = (0, 0, 0)$
Therefore we have:

$$(E) : \begin{cases} p\alpha + q\beta + 0\gamma = 0 \\ q\alpha - p\beta + 0\gamma = 0 \\ 0\alpha + 0\beta + r\gamma = 0 \end{cases}$$

Note that $\begin{vmatrix} p & q & 0 \\ q & -p & 0 \\ 0 & 0 & r \end{vmatrix} = -r(p^2 + q^2) \neq 0$
Hence (E) has unique solution only, thus $\alpha = \beta = \gamma = 0$
Therefore $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly independent

Solution (Solution to (b)).
Let $\mathbf{d} = (d_1, d_2, d_3)$, consider the system of linear equation in (α, β, γ):

$$(F) : \begin{cases} p\alpha + q\beta + 0\gamma = d_1 \\ q\alpha - p\beta + 0\gamma = d_2 \\ 0\alpha + 0\beta + r\gamma = d_3 \end{cases}$$

By (a), $\begin{vmatrix} p & q & 0 \\ q & -p & 0 \\ 0 & 0 & r \end{vmatrix} = -r(p^2 + q^2) \neq 0$
Hence (F) has unique solution, thus $\mathbf{d} = \alpha\,\mathbf{a} + \beta\,\mathbf{b} + \gamma\,\mathbf{c}$ for some $\alpha, \beta, \gamma \in \mathbb{R}$
Notice that $\langle \mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{b}, \mathbf{c} \rangle = \langle \mathbf{c}, \mathbf{a} \rangle = 0$
$\langle \mathbf{d}, \mathbf{a} \rangle = \alpha \langle \mathbf{a}, \mathbf{a} \rangle \Longrightarrow \alpha = \dfrac{\langle \mathbf{d}, \mathbf{a} \rangle}{\langle \mathbf{a}, \mathbf{a} \rangle}$ (Since $\langle \mathbf{a}, \mathbf{a} \rangle = p^2 + q^2 \neq 0$)
$\langle \mathbf{d}, \mathbf{b} \rangle = \beta \langle \mathbf{b}, \mathbf{b} \rangle \Longrightarrow \beta = \dfrac{\langle \mathbf{d}, \mathbf{b} \rangle}{\langle \mathbf{b}, \mathbf{b} \rangle}$ (Since $\langle \mathbf{b}, \mathbf{b} \rangle = q^2 + p^2 \neq 0$)
$\langle \mathbf{d}, \mathbf{c} \rangle = \gamma \langle \mathbf{c}, \mathbf{c} \rangle \Longrightarrow \gamma = \dfrac{\langle \mathbf{d}, \mathbf{c} \rangle}{\langle \mathbf{c}, \mathbf{c} \rangle}$ (Since $\langle \mathbf{c}, \mathbf{c} \rangle = r^2 \neq 0$)
Thus $\mathbf{d} = \dfrac{\langle \mathbf{d}, \mathbf{a} \rangle}{\langle \mathbf{a}, \mathbf{a} \rangle} \mathbf{a} + \dfrac{\langle \mathbf{d}, \mathbf{b} \rangle}{\langle \mathbf{b}, \mathbf{b} \rangle} \mathbf{b} + \dfrac{\langle \mathbf{d}, \mathbf{c} \rangle}{\langle \mathbf{c}, \mathbf{c} \rangle} \mathbf{c}$

Solution (Solution to (c)).

Suppose, on the contrary, $\mathbf{v} = 0$, then $\mathbf{u} = \mathbf{x}$ and $\mathbf{v} = \mathbf{y} - \dfrac{\langle \mathbf{y}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\, \mathbf{u} = 0$

By assumption, $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are linearly independent, hence $\mathbf{x} \neq \mathbf{0}$

Then $\mathbf{y} = \dfrac{\langle \mathbf{y}, \mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle}\, \mathbf{x}$

Therefore $\left(-\dfrac{\langle \mathbf{y}, \mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle} \right) \mathbf{x} + (1)\, \mathbf{y} + (0)\, \mathbf{z} = \mathbf{0}$

Since $1 \neq 0$, hence there exists some linear combination (r, s, t) such that $r\, \mathbf{x} + s\, \mathbf{y} + t\, \mathbf{z} = 0$

That is: $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are linearly dependent, contradiction !

It follows that, in first place, $\mathbf{v} \neq 0$

Solution (Solution to (d)).

$$
\begin{aligned}
\langle \mathbf{u}, \mathbf{v} \rangle &= \left\langle \mathbf{x}, \mathbf{y} - \frac{\langle \mathbf{y}, \mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle}\, \mathbf{x} \right\rangle \\
&= \langle \mathbf{x}, \mathbf{y} \rangle - \frac{\langle \mathbf{y}, \mathbf{x} \rangle \langle \mathbf{x}, \mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle} \\
&= 0 \\
\langle \mathbf{v}, \mathbf{w} \rangle &= \left\langle \mathbf{v}, \mathbf{z} - \frac{\langle \mathbf{z}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\, \mathbf{u} - \frac{\langle \mathbf{z}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\, \mathbf{v} \right\rangle \\
&= \langle \mathbf{v}, \mathbf{z} \rangle - \frac{\langle \mathbf{z}, \mathbf{u} \rangle \langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} - \frac{\langle \mathbf{z}, \mathbf{v} \rangle \langle \mathbf{v}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \\
&= \langle \mathbf{v}, \mathbf{z} \rangle - 0 - \frac{\langle \mathbf{z}, \mathbf{v} \rangle \langle \mathbf{v}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} && (\text{Since } \langle \mathbf{v}, \mathbf{u} \rangle = 0) \\
&= 0 && (\text{Since } \langle \mathbf{v}, \mathbf{z} \rangle = \langle \mathbf{z}, \mathbf{v} \rangle) \\
\langle \mathbf{u}, \mathbf{w} \rangle &= \left\langle \mathbf{u}, \mathbf{z} - \frac{\langle \mathbf{z}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\, \mathbf{u} - \frac{\langle \mathbf{z}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\, \mathbf{v} \right\rangle \\
&= \langle \mathbf{u}, \mathbf{z} \rangle - \frac{\langle \mathbf{z}, \mathbf{u} \rangle \langle \mathbf{u}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} - \frac{\langle \mathbf{z}, \mathbf{v} \rangle \langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \\
&= \langle \mathbf{u}, \mathbf{z} \rangle - \frac{\langle \mathbf{z}, \mathbf{u} \rangle \langle \mathbf{u}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} - 0 && (\text{Since } \langle \mathbf{v}, \mathbf{u} \rangle = 0) \\
&= 0 && (\text{Since } \langle \mathbf{u}, \mathbf{z} \rangle = \langle \mathbf{z}, \mathbf{u} \rangle)
\end{aligned}
$$

Hence $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are orthogonal vectors, moreover, $\mathbf{w}$ is a vector perpendicular to the plane containing $\mathbf{x}, \mathbf{y}$

3.2 Orthogonal Matrix & Isometry

Exercise 3.3 (2023 TDG Quiz 1 Q4)**.** Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.

(a). Show that $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2)$

(b). Let A be a 3×3 matrix such that for any $\mathbf{v} \in \mathbb{R}^3$, $\|A\mathbf{v}\| = \|\mathbf{v}\|$. Show that for any $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^3$,

$$\langle A\mathbf{u}, A\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$$

(c). Write down any 3×3 matrix A except $\pm I$, such that for all $\mathbf{v} \in \mathbb{R}^3$, $\|A\mathbf{v}\| = \|\mathbf{v}\|$.

Solution (Solution to (a)).

$$\begin{aligned}
RHS &= \frac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2) \\
&= \frac{1}{2}(\langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle - \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle) \\
&= \frac{1}{2}(\langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle) \\
&= \frac{1}{2}(\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle) \\
&= \langle \mathbf{u}, \mathbf{v} \rangle \\
&= LHS
\end{aligned}$$

Solution (Solution to (b)).

$$\begin{aligned}
LHS &= \langle A\,\mathbf{u}, A\,\mathbf{v} \rangle \\
&= \frac{1}{2}(\|A\,\mathbf{u}\|^2 + \|A\,\mathbf{v}\|^2 - \|A\,\mathbf{u} - A\,\mathbf{v}\|^2) \\
&= \frac{1}{2}(\|A\,\mathbf{u}\|^2 + \|A\,\mathbf{v}\|^2 - \|A(\mathbf{u} - \mathbf{v})\|^2) \\
&= \frac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2) \\
&= \langle \mathbf{u}, \mathbf{v} \rangle
\end{aligned}$$

3.3 Vector-Valued Function & Differentiation By Part

Exercise 3.4 (Lecture Notes Chapter 1 Exercise 6)**.**
Let $\mathbf{v}(t)$ be a differentiable vector-valued function. Suppose $\|\mathbf{v}\|$ is a constant independent of t, prove that $\frac{d\,\mathbf{v}}{dt}$ is orthogonal to $\mathbf{v}$ at any t

Solution. Suppose $\|\mathbf{v}(t)\| =$ constant, then $\langle \mathbf{v}(t), \mathbf{v}(t) \rangle = C$, for some $C \in \mathbb{R}$ independent of t

Differentiating both sides:

$$\begin{aligned}
\frac{d}{dt}\langle \mathbf{v}(t), \mathbf{v}(t) \rangle &= \frac{d}{dt}C \\
\langle \mathbf{v}(t), \mathbf{v}'(t) \rangle + \langle \mathbf{v}'(t), \mathbf{v}(t) \rangle &= 0 \\
2\langle \mathbf{v}'(t), \mathbf{v}(t) \rangle &= 0 \\
\langle \mathbf{v}'(t), \mathbf{v}(t) \rangle &= 0
\end{aligned}$$

Hence we have $\mathbf{v}'(t) \perp \mathbf{v}(t)$ for any t

4 Harder Question

Exercise 4.1 (Reflection in $\mathbb{R}^2$ and $\mathbb{R}^3$, Credit to Houston Tang)**.** Let $\mathbf{u} \in \mathbb{R}^n$ be a fixed unit vector, define Householder Transformation:

$$H_{\mathbf{u}} : \mathbb{R}^n \to \mathbb{R}^n \text{ by } H_{\mathbf{u}}(\mathbf{x}) \stackrel{\text{def}}{=} \mathbf{x} - 2\langle \mathbf{x}, \mathbf{u} \rangle \mathbf{u}$$

(a). Show that

- $H_{\mathbf{u}}$ is linear
- $H_{\mathbf{u}}(\mathbf{x}) = \mathbf{x} \iff \langle \mathbf{u}, \mathbf{x} \rangle = 0$
- $H_{\mathbf{u}}(\mathbf{u}) = -\mathbf{u}$
- $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \langle H_{\mathbf{u}}(\mathbf{x}), \mathbf{y} \rangle = \langle \mathbf{x}, H_{\mathbf{u}}(\mathbf{y}) \rangle$
- $H_{\mathbf{u}}$ is distance-preserving
- $H_{\mathbf{u}} \circ H_{\mathbf{u}} = \mathrm{Id}$

(b). By considering $\mathbf{u} = -(\sin\theta)\,\mathbf{i} + (\cos\theta)\,\mathbf{j}$ and $n = 2$, derive the matrix representation of $H_{\mathbf{u}}$. It should be something you familiar.

(c). Generalise your answer to higher dimensions

Solution (Solution to (a)).

(1). Pick any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, $\alpha \in \mathbb{R}$, then:

$$\begin{aligned} H_{\mathbf{u}}(\alpha\,\mathbf{x} + \mathbf{y}) &= \alpha\,\mathbf{x} + \mathbf{y} - 2\langle \alpha\,\mathbf{x} + \mathbf{y}, \mathbf{u} \rangle \mathbf{u} \\ &= \alpha\,\mathbf{x} + \mathbf{y} - 2\alpha\langle \mathbf{x}, \mathbf{u} \rangle - \langle \mathbf{y}, \mathbf{u} \rangle \mathbf{u} \\ &= \alpha(\,\mathbf{x} - 2\langle \mathbf{x}, \mathbf{u} \rangle \mathbf{u}\,) + [\,\mathbf{y} - 2\langle \mathbf{y}, \mathbf{u} \rangle \mathbf{u}\,] \\ &= \alpha H_{\mathbf{u}}(\mathbf{x}) + H_{\mathbf{u}}(\mathbf{y}) \end{aligned}$$

(2). Let $\mathbf{x} \in \mathbb{R}$

$$\begin{aligned} H_{\mathbf{u}}(\mathbf{x}) = \mathbf{x} &\iff \mathbf{x} - 2\langle \mathbf{x}, \mathbf{u} \rangle \mathbf{u} = \mathbf{x} \\ &\iff \langle \mathbf{x}, \mathbf{u} \rangle \mathbf{u} = \mathbf{0} \\ &\iff |\langle \mathbf{x}, \mathbf{u} \rangle| \|\mathbf{u}\| = 0 \\ &\iff |\langle \mathbf{x}, \mathbf{u} \rangle| = 0 \\ &\iff \langle \mathbf{x}, \mathbf{u} \rangle = 0 \end{aligned}$$

(3). $H_{\mathbf{u}}(\mathbf{u}) = \mathbf{u} - 2\langle \mathbf{u}, \mathbf{u} \rangle \mathbf{u} = \mathbf{u} - 2\,\mathbf{u} = -\,\mathbf{u}$

(4). Pick any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then:

$$\begin{aligned} \langle H_{\mathbf{u}}(\mathbf{x}), \mathbf{y} \rangle &= \langle \mathbf{x} - 2\langle \mathbf{x}, \mathbf{u} \rangle \mathbf{u}, \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{y} \rangle - 2\langle \mathbf{x}, \mathbf{u} \rangle \langle \mathbf{u}, \mathbf{y} \rangle \\ &= \langle \mathbf{y}, \mathbf{x} \rangle - 2\langle \mathbf{y}, \mathbf{u} \rangle \langle \mathbf{u}, \mathbf{x} \rangle \\ &= \langle \mathbf{y} - 2\langle \mathbf{y}, \mathbf{u} \rangle \mathbf{u}, \mathbf{x} \rangle \\ &= \langle \mathbf{x}, \mathbf{y} - 2\langle \mathbf{y}, \mathbf{u} \rangle \mathbf{u} \rangle \\ &= \langle \mathbf{x}, H_{\mathbf{u}}(\mathbf{y}) \rangle \end{aligned}$$

(5). $\langle H_{\mathbf{u}}(\mathbf{x}), H_{\mathbf{u}}(\mathbf{x}) \rangle = \langle \mathbf{x} - 2\langle \mathbf{x}, \mathbf{u} \rangle \mathbf{u}, \mathbf{x} - 2\langle \mathbf{x}, \mathbf{u} \rangle \mathbf{u} \rangle = \|\mathbf{x}\|^2 - 4\langle \mathbf{u}, \mathbf{x} \rangle^2 + 4\langle \mathbf{u}, \mathbf{x} \rangle^2 \langle \mathbf{u}, \mathbf{u} \rangle = \|\mathbf{x}\|^2$

(6). Let $\mathbf{x} \in \mathbb{R}^3$, then:

$$\begin{aligned} H_{\mathbf{u}}(H_{\mathbf{u}}(\mathbf{x})) &= H_{\mathbf{u}}(\,\mathbf{x} - 2\langle \mathbf{x}, \mathbf{u}\rangle\, \mathbf{u}\,) \\ &= \mathbf{x} - 2\langle \mathbf{x}, \mathbf{u}\rangle\, \mathbf{u} - 2\langle \mathbf{x} - 2\langle \mathbf{x}, \mathbf{u}\rangle\, \mathbf{u}, \mathbf{u}\rangle\, \mathbf{u} \\ &= \mathbf{x} - 2\langle \mathbf{x}, \mathbf{u}\rangle\, \mathbf{u} - 2\langle \mathbf{x}, \mathbf{u}\rangle\, \mathbf{u} + 4\langle \mathbf{x}, \mathbf{u}\rangle\langle \mathbf{u}, \mathbf{u}\rangle\, \mathbf{u} \\ &= \mathbf{x} - 2\langle \mathbf{x}, \mathbf{u}\rangle\, \mathbf{u} - 2\langle \mathbf{x}, \mathbf{u}\rangle\, \mathbf{u} + 4\langle \mathbf{x}, \mathbf{u}\rangle\, \mathbf{u} \\ &= \mathbf{x} \end{aligned}$$

Solution (Solution to (b)). Consider $\mathbf{u} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$

$$\begin{aligned} H_{\mathbf{u}}\left[\begin{pmatrix}1\\0\end{pmatrix}\right] &= \begin{pmatrix}1\\0\end{pmatrix} - 2\left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\right\rangle \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} \\ &= \begin{pmatrix}1\\0\end{pmatrix} - 2(-\sin\theta)\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} \\ &= \begin{pmatrix}1-2\sin^2\theta\\ 2\sin\theta\cos\theta\end{pmatrix} \\ &= \begin{pmatrix}\cos 2\theta\\ \sin 2\theta\end{pmatrix} \end{aligned}$$

$$\begin{aligned} H_{\mathbf{u}}\left[\begin{pmatrix}0\\1\end{pmatrix}\right] &= \begin{pmatrix}0\\1\end{pmatrix} - 2\left\langle \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\right\rangle \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} \\ &= \begin{pmatrix}0\\1\end{pmatrix} - 2(\cos\theta)\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} \\ &= \begin{pmatrix}2\sin\theta\cos\theta\\ 1-2\cos^2\theta\end{pmatrix} \\ &= \begin{pmatrix}\sin 2\theta\\ -\cos 2\theta\end{pmatrix} \end{aligned}$$

Hence matrix representation of $H_{\mathbf{u}}$ is: ($\mathbb{R}^2$ reflection matrix)

$$\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}$$

Exercise 4.2 (Bessel's Inequality)**.** Denote $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Let $\mathbf{x} \in \mathbb{R}^m$ be any arbitrary vector, where $m \geq n$. Show that:

$$\|\mathbf{x}\| \geq \left\| \mathbf{x} - \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \, \mathbf{v}_i \right\|$$

Solution.

$$\begin{aligned}
RHS^2 &= \left\| \mathbf{x} - \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \, \mathbf{v}_i \right\|^2 \\
&= \left\langle \mathbf{x} - \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \, \mathbf{v}_i, \mathbf{x} - \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \, \mathbf{v}_i \right\rangle \\
&= \|\mathbf{x}\|^2 - 2\left\langle \mathbf{x}, \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \, \mathbf{v}_i \right\rangle + \left\langle \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \, \mathbf{v}_i, \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{v}_j \rangle \, \mathbf{v}_j \right\rangle \\
&= \|\mathbf{x}\|^2 - 2\sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \langle \mathbf{x}, \mathbf{v}_i \rangle + \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \langle \mathbf{x}, \mathbf{v}_j \rangle \underbrace{\langle \mathbf{v}_i, \mathbf{v}_j \rangle}_{\text{zero if } i \neq j} \\
&= \|\mathbf{x}\|^2 - 2\sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \langle \mathbf{x}, \mathbf{v}_i \rangle + \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \langle \mathbf{x}, \mathbf{v}_i \rangle \langle \mathbf{v}_i, \mathbf{v}_i \rangle \\
&= \|\mathbf{x}\|^2 - 2\sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \langle \mathbf{x}, \mathbf{v}_i \rangle + \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \langle \mathbf{x}, \mathbf{v}_i \rangle \\
&= \|\mathbf{x}\|^2 - \sum_{i=1}^{n} \left[\langle \mathbf{x}, \mathbf{v}_i \rangle \right]^2 \\
&\leq \|\mathbf{x}\|^2 \\
&= LHS^2
\end{aligned}$$

5 Challenging Question

Exercise 5.1 (Lecture Notes Chapter 2 Exercise 11 Modified)**.**
Let $\mathbf{r}(s)$ be a differentiable vector-valued function on $\mathbb{R}^2$ with $\|\mathbf{r}'(s)\|= 1$. Denote $\mathbf{T}(s) = \mathbf{r}'(s)$

(a). Show that $\langle \mathbf{T}'(s), \mathbf{T}(s)\rangle = 0$

(b). Denote $\kappa(s) \stackrel{\text{def}}{=} \|\mathbf{T}'(s)\|$, define $\mathbf{N}(s)$ by the relation: $\mathbf{T}'(s) = \kappa(s)\,\mathbf{N}(s)$
Compute $\|\mathbf{N}(s)\|$ and $\langle \mathbf{T}(s), \mathbf{N}(s)\rangle$, deduce that $\langle \mathbf{T}(s), \mathbf{N}'(s)\rangle = -\kappa(s)$

(c). Does $\{\mathbf{T}(s), \mathbf{N}(s)\}$ constitute a orthonormal basis for $\mathbb{R}^2$? Hence prove or disprove:

$$\mathbf{N}'(s) = -\kappa(s)\,\mathbf{T}(s)$$

Solution (Solution to (a)). By assumption, $\|\mathbf{r}'(s)\|= 1$ and $\mathbf{T}(s) = \mathbf{r}'(s)$, hence $\|\mathbf{T}(s)\|= 1$

$$\begin{aligned}
\langle \mathbf{T}(s), \mathbf{T}(s)\rangle &= 1\\
\frac{d}{ds}\langle \mathbf{T}(s), \mathbf{T}(s)\rangle &= \frac{d}{ds}(1)\\
\langle \mathbf{T}'(s), \mathbf{T}(s)\rangle + \langle \mathbf{T}(s), \mathbf{T}'(s)\rangle &= 0\\
2\langle \mathbf{T}'(s), \mathbf{T}(s)\rangle &= 0\\
\langle \mathbf{T}'(s), \mathbf{T}(s)\rangle &= 0
\end{aligned}$$

Solution (Solution to (b)).

(1). By assumption, $\mathbf{T}'(s) = \kappa(s)\,\mathbf{N}(s)$, hence:

$$\begin{aligned}
\|\mathbf{T}'(s)\| &= \|\kappa(s)\,\mathbf{N}(s)\|\\
\kappa(s) &= \|\kappa(s)\,\mathbf{N}(s)\|\\
\kappa(s) &= \kappa(s)\|\mathbf{N}(s)\|\\
\|\mathbf{N}(s)\| &= 1
\end{aligned}$$

(2). By part (a), $\langle \mathbf{T}'(s), \mathbf{T}(s)\rangle = 0$ and definition: $\mathbf{T}'(s) = \kappa(s)\,\mathbf{N}(s)$, hence:

$$\begin{aligned}
\langle \mathbf{T}(s), \mathbf{N}(s)\rangle &= \langle \mathbf{T}(s), \kappa(s)\,\mathbf{T}'(s)\rangle\\
&= \kappa(s)\langle \mathbf{T}'(s), \mathbf{T}(s)\rangle\\
&= \kappa(s) \times 0\\
&= 0
\end{aligned}$$

Moreover, differentiating $\langle \mathbf{T}(s), \mathbf{N}(s)\rangle = 0$ on the both side, we have:

$$\begin{aligned}
\frac{d}{ds}\langle \mathbf{T}(s), \mathbf{N}(s)\rangle &= \frac{d}{ds}(0)\\
\langle \mathbf{T}'(s), \mathbf{N}(s)\rangle + \langle \mathbf{T}(s), \mathbf{N}'(s)\rangle &= 0\\
\langle \kappa(s)\,\mathbf{N}(s), \mathbf{N}(s)\rangle + \langle \mathbf{T}(s), \mathbf{N}'(s)\rangle &= 0\\
\kappa(s)\langle \mathbf{N}(s), \mathbf{N}(s)\rangle + \langle \mathbf{T}(s), \mathbf{N}'(s)\rangle &= 0\\
\kappa(s)\|\mathbf{N}(s)\| + \langle \mathbf{T}(s), \mathbf{N}'(s)\rangle &= 0\\
\kappa(s) + \langle \mathbf{T}(s), \mathbf{N}'(s)\rangle &= 0\\
\langle \mathbf{T}(s), \mathbf{N}'(s)\rangle &= -\kappa(s)
\end{aligned}$$

Lemma 5.1. Claim: $\langle \mathbf{N}'(s), \mathbf{N}(s) \rangle = 0$

Proof: Note that $\|\mathbf{N}(s)\| = 1 \overset{d/ds}{\Longrightarrow} \langle \mathbf{N}(s), \mathbf{N}'(s) \rangle + \langle \mathbf{N}'(s), \mathbf{N}(s) \rangle = 0 \Rightarrow \langle \mathbf{N}(s), \mathbf{N}'(s) \rangle = 0$

Solution (Solution to (c)).

$\{\mathbf{T}(s), \mathbf{N}(s)\}$ constitutes an orthonormal basis since $\langle \mathbf{T}(s), \mathbf{N}(s) \rangle = 0$ and $\|\mathbf{T}(s)\| = \|\mathbf{N}(s)\| = 1$

By Tutorial 2 Theorem 2.9, suppose $\mathbf{N}'(s) = \alpha\,\mathbf{T}(s) + \beta\,\mathbf{N}(s)$, then:

$$\begin{aligned} \mathbf{N}'(s) &= \langle \mathbf{N}'(s), \mathbf{T}(s) \rangle\, \mathbf{T}(s) + \langle \mathbf{N}'(s), \mathbf{N}(s) \rangle\, \mathbf{N}(s) \\ &= -\kappa(s)\,\mathbf{T}(s) + 0 \times \mathbf{N}(s) \\ &= -\kappa(s)\,\mathbf{T}(s) \end{aligned}$$